Last time: classification of finite nilpotent groups.

$G$ is nilpotent $\Longrightarrow$ $G$ has the normalizer property
$H < G \Rightarrow H < N_G(H)$

if $G$ is finite $\Uparrow$ $\qquad\qquad$ $\Swarrow$ if $G$ is finite

$$G \cong P_1 \times \dots \times P_k$$

for finite $p$-groups $P_1, \dots, P_k$ w.r.t. distinct primes

As a tool, we used alternative description of nilpotent groups as those $G$ which possess a central series, i.e.

$$1 = G_0 \lhd G_1 \lhd \dots \lhd G_{k-1} \lhd G_k = G$$

with $G_i \lhd G$, $\forall i$ and $G_i/G_{i-1} < Z(G/G_{i-1})$, $\forall i$.

Today: beyond $\mathbb{Z}/n\mathbb{Z}$, $D_{2n}$, $S_n$, what other groups can we construct? All of them, using the tools from today.

Let's define free groups:

· consider a set $S$ called an alphabet
$$S = \{\ldots\ldots, s, \ldots\ldots\}$$
letters

· a word is a finite sequence $s_1^{\pm 1} s_2^{\pm 1} \ldots\ldots s_k^{\pm 1}$
($s_1, \ldots, s_k \in S$ , $s^{+1} = s$ , $s^{-1}$ another formal symbol)

· a reduced word is a word with does not contain $\ldots\ldots ss^{-1} \ldots\ldots$ or $\ldots\ldots s^{-1}s \ldots\ldots$ , $\forall s \in S$

**Def:** the free group corresponding to $S$ is

$$F_S = \{ \text{reduced words } s_1^{\pm 1} s_2^{\pm 1} \ldots\ldots s_k^{\pm 1} , \forall s_i \in S \}$$

· identity = empty word of length 0

· inverses $\quad (s_1^{\pm 1} \ldots\ldots s_k^{\pm 1})^{-1} = s_k^{\mp 1} \ldots\ldots s_1^{\mp}$

· operation given by concatenation + reduction

$$(s_1^{\pm 1} \ldots s_k^{\pm 1}) \cdot (t_1^{\pm 1} \ldots t_l^{\pm 1}) = s_1^{\pm 1} \ldots s_k^{\pm 1} t_1^{\pm 1} \ldots t_k^{\pm 1}$$

reduction : $\begin{array}{l} \ldots x s s^{-1} y \ldots \\ \ldots x s^{-1} s y \ldots \end{array} \rightsquigarrow \begin{array}{l} \ldots x y \ldots \\ \ldots x y \ldots \end{array}$

- $\forall$ function $f : S \longrightarrow S'$

$$\rightsquigarrow$$

homomorphism $\bar{f} : F_S \longrightarrow F_{S'}$

$$s_1^{\pm 1} \cdots s_k^{\pm 1} \rightsquigarrow f(s_1)^{\pm 1} \cdots f(s_k)^{\pm 1}$$

**Lemma:** $\forall$ set $S$, $\forall$ group $G$, $\exists$ 1-to-1 correspondence

$$\{\text{functions } f : S \longrightarrow G\} \xleftrightarrow{\Psi_{S,G}} \{\text{homomorphism } F_S \to G\}$$

which is functorial in the sense that the following diagram commutes

"natural"

$$\begin{tikzcd}
\{S \xrightarrow{\alpha} G\} \arrow[r, leftrightarrow, "\Psi_{S,G}"] \arrow[d, "\alpha \rightsquigarrow g \circ \alpha \circ f"'] & \{\text{homs } F_S \xrightarrow{\beta} G\} \arrow[d, "\beta \rightsquigarrow g \circ \beta \circ \bar{f}"] \\
\{S' \xrightarrow{\alpha'} G'\} \arrow[r, leftrightarrow, "\Psi_{S',G'}"] & \{\text{homs } F_{S'} \xrightarrow{\beta'} G'\}
\end{tikzcd}$$

$\forall$ function $S' \xrightarrow{f} S$, $\forall$ group homomorphism $G \xrightarrow{g} G'$.

Proof: $\{S \xrightarrow{\alpha} G\} \rightsquigarrow \{\text{homs } F_S \xrightarrow{\beta} G\}$

$$\beta(s_1^{\pm1}\cdots s_k^{\pm1}) = \beta(s_1)^{\pm1}\cdots\beta(s_k)^{\pm1}$$
$$:= \alpha(s_1)^{\pm1}\cdots\alpha(s_k)^{\pm1}$$

multiplication as in G

$$\{S \xrightarrow{\alpha} G\} \leftsquigarrow \{\text{homs } F_S \xrightarrow{\beta} G\}$$

$$\alpha(s) = \beta(\text{one-letter word } s)$$

Functoriality:

$\alpha \in$ $\qquad \beta(s_1^{\pm1}\cdots s_k^{\pm1}) = \alpha(s_1)^{\pm1}\cdots\alpha(s_k)^{\pm1}$

$$\begin{tikzcd}
\{S \xrightarrow{\alpha} G\} \arrow[r, leftrightarrow, "\Psi_{S,G}"] \arrow[d] & \{\text{homs } F_S \xrightarrow{\beta} G\} \arrow[d] \\
\{S' \xrightarrow{\alpha'} G'\} \arrow[r, leftrightarrow, "\Psi_{S',G'}"] & \{\text{homs } F_{S'} \xrightarrow{\beta'} G'\}
\end{tikzcd}$$

$\alpha'(s') = g(\alpha(f(s')))$

$$\beta'(s_1'^{\pm1}\cdots s_k'^{\pm1}) = g(\alpha(f(s_1')))^{\pm1}\cdots g(\alpha(f(s_k')))^{\pm1}$$

$$\beta'(s_1'^{\pm1}\cdots s_k'^{\pm1}) = g\circ\beta\circ\bar{f}(s_1'^{\pm1}\cdots s_k'^{\pm1}) = g\circ\beta(f(s_1')^{\pm1}\cdots f(s_k')^{\pm1}) = g(\alpha(f(s_1'))^{\pm1}\cdots\alpha(f(s_k'))^{\pm1})$$

$g(\alpha(f(s_1')))^{\pm 1} \cdots g(\alpha(f(s_k')))^{\pm 1}$

The compositions $\curvearrowright$ and $\hookrightarrow$ produce the same result. □

**Thm**: $S \xleftrightarrow{\text{bijection}} T \iff F_S \overset{\text{iso}}{\cong} F_T$

"$\Longrightarrow$" obvious

"$\Longleftarrow$" hard

**Def**: the **free abelian group** corresponding to $S$ is

$$F_S^{ab} = F_S / [F_S, F_S]$$

**Prop**: $F_S^{ab} \cong \mathbb{Z}^S := \bigoplus_{s \in S} \mathbb{Z} \cdot s$

group w.r.t component-wise addition

$F_S / [F_S, F_S]$

$$= \{ (\dots n_s \dots)_{s \in S}, \text{ only finitely many } n_s \neq 0 \}, \quad n_s \in \mathbb{Z}$$

$$= \{ \sum_{s \in S} n_s \cdot s, \text{ only finitely many } n_s \neq 0 \}$$

$$[s_1^{\pm 1} \cdots s_k^{\pm 1}] \rightsquigarrow \sum_{i=1}^{k} \pm s_i$$

(e.g. $x^2 y^{-3} x^{-5} y^7 \rightsquigarrow 2x - 3y - 5x + 7y = -3x + 4y$)

The assignment $\rightsquigarrow$ yields a well-defined hom $F_S^{ab} \to \mathbb{Z}^S$ because

$$\left[(s_1^{\pm1}\dots s_k^{\pm1})(t_1^{\pm1}\dots t_L^{\pm1})(s_k^{\mp1}\dots s_1^{\mp1})(t_L^{\mp1}\dots t_1^{\mp1})\right] \rightsquigarrow \begin{matrix} \pm s_1 \pm\dots\pm s_k \pm t_1 \pm\dots\pm t_L \\ \mp s_1\mp\dots\mp s_k \mp t_1\mp\dots\mp t_L\end{matrix} = 0$$

So $F_S \cong F_T \Rightarrow F_S^{ab} \cong F_T^{ab} \Rightarrow \mathbb{Z}^S \cong \mathbb{Z}^T$

$\Rightarrow (\mathbb{Z}/2\mathbb{Z})^S \cong (\mathbb{Z}/2\mathbb{Z})^T \Rightarrow \{\text{finite subset of } S\} \leftrightarrow \{\text{finite subsets of } T\}$

the same underlying set

$\Downarrow$ proved in notes

$S \xleftrightarrow{\text{bij}} T$ $\square$

- Examples: $S = \emptyset$, $F_S = 1$

  $S = \{x\}$, $F_S = \{x^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$

  $S = \{x, y\}$, $F_S$ huge but countable

  $\vdots$

**Def:** $S$ alphabet

$R = \{\text{subset of reduced words in } S\}$

$\langle S \mid R \rangle = F_S \Big/ ($

$$\langle \underbrace{S}_{\text{generators}} \mid \underbrace{R}_{\text{relations}} \rangle = F_S \Big/ \left( \begin{array}{c} \text{smallest normal subgroup of} \\ F_S \text{ which contains } R \end{array} \right)$$

$$\left\{ \begin{array}{l} \text{products of } g r g^{-1} \\ \text{and their inverses} \end{array} \;\middle|\; g \in F_S,\ r \in R \right\}$$

- Examples:
  - $R = \{ a b a^{-1} b^{-1} \mid \forall \text{ reduced } a, b \}$

    $$\langle S \mid R \rangle = F_S^{ab}$$

  - $\mathbb{Z}/_{n\mathbb{Z}} = \langle x \mid x^n \rangle$

  - $D_{2n} = \langle \sigma, \tau \mid \sigma^n, \tau^2, (\sigma\tau)^2 \rangle$

  - $S_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i^2, (\sigma_i \sigma_{i+1})^3, (\sigma_i \sigma_j)^2 \rangle_{\forall |i-j|>1}$

    transpositions $(12), \dots, (n-1\ n)$

A group $G$ is called

- **finitely generated** if $G \cong \langle S \mid R \rangle$ for some $|S| < \infty$
- **finitely presented** if $G \cong \langle S \mid R \rangle$ for some $|S|, |R| < \infty$

It may happen that $\langle S \mid R \rangle \cong \langle S' \mid R' \rangle$ for different

$(S,R)$ and $(S',R')$ ; so the same group can be presented by generators and relations in many different ways, e.g.

$$\underbrace{\langle x,y \mid x^7y^9 = y^{-4}x^5,\ x^6 = e \rangle}_{= \langle x,y \mid x^7y^9x^{-5}y^4,\ x^6 \rangle} \cong \langle x,y,z \mid x^7y^9x^{-5}y^4,\ x^6, z \rangle$$

extra generator      extra relation

**Thm:** $\forall$ group $G$ can be presented by generators and relations

$$G \cong \langle S \mid R \rangle \quad \text{for some } S, R$$

<u>Proof</u>: $S = G$ , letters will be $\bar{g}$ , $\forall g \in G$

$$R = \{ \bar{g}\bar{h} = \overline{gh},\ \bar{g}^{-1} = \overline{g^{-1}},\ \bar{e} \}$$

(least economically possible, relations are not independent)

**Lemma:** $\forall$ alphabet $S$, $\forall$ set of words $R$, $\forall$ group $G$,

$\exists$ a 1-to-1 correspondence

$$\{ \quad R\} \overset{\Psi_{S|R,G}}{\quad} \{ \quad \langle S \mid R \rangle \to G \}$$

$$\left\{ S \xrightarrow{\alpha} G \mid \alpha(r) = e, \forall r \in R \right\} \longleftrightarrow \left\{ \text{homs } \langle S | R \rangle \to G \right\}$$

which is functorial, i.e. the following diagram commutes

$$\begin{tikzcd}
\left\{ S \xrightarrow{\alpha} G \mid \alpha(r) = e, \forall r \in R \right\} \arrow[d] & \left\{ \text{homs } \langle S | R \rangle \to G \right\} \arrow[l, "\Psi_{S|R,G}"'] \arrow[d] \\
\left\{ S' \xrightarrow{\alpha'} G' \mid \alpha(r') = e, \forall r' \in R' \right\} & \left\{ \text{homs } \langle S' | R' \rangle \to G' \right\} \arrow[l, "\Psi_{S'|R',G'}"']
\end{tikzcd}$$

$\forall$ function $S' \xrightarrow{f} S$, $\forall$ hom $G \xrightarrow{g} G'$ s.t. $f(R') \subset$ concatenations of words in $R$ and their inverses

$\square$

Lemma above actually provides a universal property that defines $\langle S | R \rangle$, i.e.

Even if you didn't know the explicit construction of $\langle S | R \rangle$, $\exists$ a unique (up to isomorphism) group $\langle S | R \rangle$ which satisfies conditions in the Lemma.

Proof: suppose $\exists$ $\langle S | R \rangle$ and $\langle S | R \rangle'$ satisfying Lemma; then

$\exists$ 1-to-1 $\{\text{homs } \langle S|R\rangle \longrightarrow G\} \overset{\Phi_G}{\longleftrightarrow} \{\text{homs } \langle S|R\rangle' \longrightarrow G\}$

functorial

take $G = \langle S|R\rangle$ : $\left(Id : \langle S|R\rangle \to \langle S|R\rangle\right) \rightsquigarrow \left(\langle S|R\rangle' \xrightarrow{\beta'} \langle S|R\rangle\right)$

take $G = \langle S|R\rangle'$ : $\left(\langle S|R\rangle \xrightarrow{\beta} \langle S|R\rangle'\right) \leftsquigarrow \left(Id : \langle S|R\rangle' \to \langle S|R\rangle'\right)$

Lecture notes : functoriality implies $\beta \circ \beta' = \beta' \circ \beta = Id$

$$\Downarrow$$

$$\langle S|R\rangle \cong \langle S|R\rangle'$$